

Fast and robust two- and three-qubit swapping gates on multi-atomic ensembles in quantum electrodynamic cavity

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Qubits on multi-atomic ensembles in a common optical resonator are considered. With that, possible constructions of swap, square root of swap and controlled swap quantum gates are analyzed. Dynamical elimination of excess quantum state and collective blockade mechanism are proposed for realizations of the two and three qubit gates.

1 Introduction

The creation of a quantum computer is an outstanding fundamental and practical problem. The quantum computer could be used for the execution of very complicated tasks which are not solvable with the classical computers. The first prototype of a solid state quantum computer was created in 2009 with superconducting qubits [1]. However, it suffers from the decoherent processes and it is desirable to find more practical encoding of qubits with long-lived coherence. It could be single impurity or vacancy centers in solids [2] but their interaction with electromagnetic radiation is rather weak. So, here, ensembles of atoms were proposed for the qubit encoding by using the dipole blockade mechanism in order to turn multilevel systems in two level ones [3]. But dipole-dipole based blockade introduces an additional decoherence that limits its practical significance. Recently, the collective blockade mechanism has been proposed for the system of three-level atoms by using the different frequency shifts for the Raman transitions between the collective atomic states characterized by a different number of the excited atoms [4]. Here, we propose a two qubit gate by using another collective blockade mechanism in the system of two level atoms based on exchange interaction via the virtual photons between the multi-atomic ensembles in the resonator. Also we demonstrate the possibility of a three qubit gate (Controlled SWAP gate) using a suppression of the swap-process between two multi-atomic ensembles due to dynamical shift of the atomic levels controlled by the states of photon encoded qubit.

2 Swap gates

Let us consider a plurality of the atomic systems (nodes) situating in a common electrodynamic resonator with a quantum memory (QM) node as depicted in Fig. 1. For realization of two-qubit gates we transfer the two qubits from QM node to the 1-st and 2-rd processing nodes and equalize the carrier frequencies

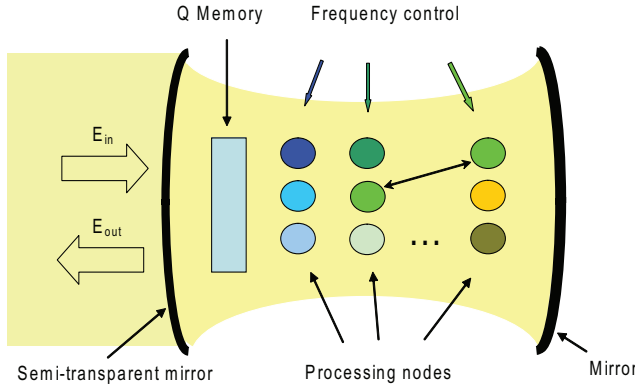


Figure 1: Layout for realization of two-qubit *i*SWAP gates. Input and output quantum information is encoded through the quantum states of the in-coming (E_{in}) and out-coming (E_{out}) single photon fields.

of the nodes at time moment $t=0$ with some detuning from the resonator mode frequency $\omega_1 - \omega_0 = \Delta_1 = \omega_2 - \omega_0 = \Delta_2 = \Delta$. It yields to the following initial state of 1-st and 2-nd nodes in the interaction picture

$$\psi_{in}(0) = \{\alpha_1|0\rangle_1 + \beta_1|1\rangle_1\}\{\alpha_2|0\rangle_2 + \beta_2|1\rangle_2\}, \quad (1)$$

where $|\alpha_{1,2}|^2 + |\beta_{1,2}|^2 = 1$. Here, we have introduced the following states: $|0\rangle_m = |0_1, 0_2, \dots, 0_{N_m}\rangle$ corresponding to the ground state of the m -th node, $|1\rangle_m = \sqrt{1/N} \sum_j^{N_m} |0\rangle_1 |0\rangle_2 \dots |1\rangle_j \dots |0\rangle_{N_m}$ and $|2\rangle_m = \sqrt{2/N(N-1)} \sum_{i \neq j}^{N_m} |0\rangle_1 |0\rangle_2 \dots |1\rangle_i \dots |1\rangle_j \dots |0\rangle_{N_m}$ are the collective states of m -th node with single and two atomic excitations. Equal frequencies of the two nodes results in interaction of the atoms via the virtual processes of resonant circuit quanta determined by effective Hamiltonian^{1,2} between these nodes. In order to obtain this interaction, we start from initial Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$ where $\hat{H}_0 = \hat{H}_a + \hat{H}_r$ is main Hamiltonian and $\hat{H}_1 = \hat{H}_{r-a}$ is perturbation Hamiltonian. Here, $\hat{H}_a = \hat{H}_{a_1} + \hat{H}_{a_2}$ is Hamiltonian of atoms in nodes 1 and 2 and \hat{H}_r is Hamiltonian of photons. With that, $\hat{H}_{a_1} = \hbar\omega_0 \sum_{j_1} S_{j_1}^z$ and $\hat{H}_{a_2} = \hbar\omega_0 \sum_{j_2} S_{j_2}^z$ where ω_0 is the frequency of working transitions in atoms, $S_{j_1}^z$ and $S_{j_2}^z$ operators of effective spin z -projection in two-level model for atoms in sites j_1 and j_2 of nodes 1 and 2; $\hat{H}_r = \hbar\omega_{k_0} a_{k_0}^\dagger a_{k_0}$ where ω_{k_0} is frequency of photons with wave vector k_0 , $a_{k_0}^\dagger$ and a_{k_0} are creation and annihilation operators for photons. We have for interaction of atoms with photons $\hat{H}_{r-a} = H_{r-a}^{(1)} + H_{r-a}^{(2)}$ in nodes 1 and 2 the following expressions:

$$H_{r-a}^{(1)} = \sum_{j_1} (g_{k_0}^{(1)} e^{ik_0 r_{j_1}} S_{j_1}^+ a_{k_0} + g_{k_0}^{(1)*} e^{-ik_0 r_{j_1}} S_{j_1}^- a_{k_0}^\dagger), \quad (2)$$

$$H_{r-a}^{(2)} = \sum_{j_2} (g_{k_0}^{(2)} e^{ik_0 r_{j_2}} S_{j_2}^+ a_{k_0} + g_{k_0}^{(2)*} e^{-ik_0 r_{j_2}} S_{j_2}^- a_{k_0}^\dagger), \quad (3)$$

where $g_{k_0}^{(\alpha)}$ are interaction constants, $S_{j_2}^\pm$ are raising and lowering operators for spin $1/2$ in two level model, \vec{r}_{j_α} are radius vectors for atoms in sites j_α of nodes $\alpha = 1, 2$.

We perform unitary transformation of Hamiltonian $H_s = e^{-s} H e^s$ that yields in the second degree on small perturbation the following result:

$$H_s = H_0 + \frac{1}{2} [H_1, s], \quad (4)$$

when relation $H_1 + [H_0, s] = 0$ is valid. Using relation (3) we find $s = s_1 + s_2$,

$$s_1 = \sum_{j_1} \left(\alpha_1 g_{k_0}^{(1)} e^{i\vec{k}_0 \vec{r}_{j_1}} S_{j_1}^+ a_{k_0} + \beta_1 g_{k_0}^{(1)*} e^{-i\vec{k}_0 \vec{r}_{j_1}} S_{j_1}^- a_{k_0}^+ \right), \quad (5)$$

$$s_2 = \sum_{j_2} \left(\alpha_2 g_{k_0}^{(2)} e^{i\vec{k}_0 \vec{r}_{j_2}} S_{j_2}^+ a_{k_0} + \beta_2 g_{k_0}^{(2)*} e^{-i\vec{k}_0 \vec{r}_{j_2}} S_{j_2}^- a_{k_0}^+ \right), \quad (6)$$

where

$$\alpha_{1,2} = -\beta_{1,2} = -\frac{1}{\hbar(\omega_0 - \omega_{k_0})} = -\frac{1}{\hbar\Delta}. \quad (7)$$

Substituting (5) and (6) into (4), we get

$$\begin{aligned} H_s = & \hbar\omega_{k_0} a_{k_0}^+ a_{k_0} + \sum_m \sum_{j_m} \hbar\omega_m S_{j_m}^z + 2 \sum_m \sum_{j_m} \frac{|g_{k_0}^{(m)}|^2}{\hbar\Delta} a_{k_0}^+ a_{k_0} S_{j_m}^z + \sum_m \sum_{i_m j_m} \frac{|g_{k_0}^{(m)}|^2}{\hbar\Delta} S_{i_m}^+ S_{j_m}^- + \\ & + \frac{1}{\hbar\Delta} \sum_{j_1 j_2} \left(g_{k_0}^{(1)} g_{k_0}^{(2)*} e^{i\vec{k}_0 \vec{r}_{j_1 j_2}} S_{j_1}^+ S_{j_2}^- + g_{k_0}^{(1)*} g_{k_0}^{(2)} e^{-i\vec{k}_0 \vec{r}_{j_1 j_2}} S_{j_1}^- S_{j_2}^+ \right). \end{aligned} \quad (8)$$

The first term is unchanged energy of photons, the second term is unchanged energy of atoms in nodes 1 and 2, the third term is atomic energy shifts due to photons, the forth term is atomic intra-node swap energy, the fifth term is atomic inter-node swap energy.

According to (8) effective interaction of atoms is $\hat{H}_{eff} = \sum_{m=1}^2 \hat{H}_{node}^{(m)} + \hat{H}_{int}$, where $\hat{H}_{node}^{(m)} = \hbar\Omega_\sigma \sum_{i_m j_m}^N e^{i\vec{k}_0 \vec{r}_{i_m j_m}} S_{i_m}^+ S_{j_m}^-$ is a long-range spin-spin interaction in m-th node, $\hat{H}_{int} = \hbar\Omega_\sigma \sum_{j_1, j_2=1}^N \left(e^{i\vec{k}_0 \vec{r}_{j_1 j_2}} S_{j_1}^+ S_{j_2}^- + e^{-i\vec{k}_0 \vec{r}_{j_1 j_2}} S_{j_1}^- S_{j_2}^+ \right)$ (where $\Omega_\sigma = |g_\sigma|^2/\Delta$) describes a spin-spin interaction between the two nodes ($N_1 = N_2 = N$), \vec{k}_0 is wave vector of resonant mode. Let's introduce the collective basis states of the two nodes: $|\psi\rangle_1 = |0\rangle_1 |0\rangle_2$, $|\psi\rangle_2 = |1\rangle_1 |0\rangle_2$, $|\psi\rangle_3 = |0\rangle_1 |1\rangle_2$, $|\psi\rangle_4 = |1\rangle_1 |1\rangle_2$ and $|\psi\rangle_5 = 1/\sqrt{2}\{|2\rangle_1 |0\rangle_2 + |0\rangle_1 |2\rangle_2\}$. It is important that the Hamiltonian \hat{H}_{eff} has a matrix representation in the basis of the five states which is separated from other states of the multi-atomic system

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & N\Omega_\sigma & N\Omega_\sigma & 0 & 0 \\ 0 & N\Omega_\sigma & N\Omega_\sigma & 0 & 0 \\ 0 & 0 & 0 & 2N\Omega_\sigma & 2\Omega_\sigma\sqrt{N(N-1)} \\ 0 & 0 & 0 & 2\Omega_\sigma\sqrt{N(N-1)} & 2\Omega_\sigma(N-1) \end{pmatrix}. \quad (9)$$

By using (9), we find the unitary evolution of the atomic systems which couples independently two pairs of the quantum states $|\psi\rangle_2 \leftrightarrow |\psi\rangle_3$ and $|\psi\rangle_4 \leftrightarrow |\psi\rangle_5$

$$\begin{aligned}
\Psi_1(t) = & \alpha_2 \alpha_3 \psi_1 + \exp(-i\Omega_\sigma Nt) \{ \beta_2 \alpha_3 [\cos(\Omega_\sigma Nt) \psi_2 - i \sin(\Omega_\sigma Nt) \psi_3] \\
& + \alpha_2 \beta_3 [\cos(\Omega_\sigma Nt) \psi_3 - i \sin(\Omega_\sigma Nt) \psi_2] \} \\
& + \exp(-i2\Omega_\sigma Nt) \beta_2 \beta_3 [\cos(2\Omega_\sigma Nt) \psi_4 - i \sin(2\Omega_\sigma Nt) \psi_5],
\end{aligned} \tag{10}$$

where we have assumed a large number of atoms $N \gg 1$. The solution demonstrates two coherent oscillations with the frequency $\Omega_\sigma N$ for the first pair $|\psi\rangle_2 \leftrightarrow |\psi\rangle_3$ and with the double frequency $2\Omega_\sigma N$ for the second pair $|\psi\rangle_4 \leftrightarrow |\psi\rangle_5$. The oscillations are drastically accelerated N -times comparing to the case of two coupled two-level atoms so we can use even bad common resonator with relatively lower quality factor.

It is known [4,5] that the evolution of the two coupled two level atoms can lead to *iSWAP* and \sqrt{iSWAP} gates. The *iSWAP* and \sqrt{iSWAP} gates work in the Hilbert space of four states $|\psi\rangle_1, \dots, |\psi\rangle_4$ and these gates are important for realization of the complete set of the universal quantum gates [5, 6]. *iSWAP* gate provides exchange of the two quantum states between the two nodes. In our case we get that *iSWAP* gate occurs at shortened time $t_{iSWAP} = \pi/2\Omega_\sigma N$ sec

$$\Psi_1(t_{iSWAP}) = \{ \alpha_2 |0\rangle_1 - \beta_2 |1\rangle_1 \} \{ \alpha_1 |0\rangle_2 - \beta_1 |1\rangle_2 \}. \tag{11}$$

We also note that by choosing different carrier frequencies we can realize the described *iSWAP* operation for many pairs of nodes simultaneously due to exploitation of the independent virtual quanta for each pair in the QED cavity. It is interesting that the *iSWAP* gate provides a perfect elimination of transfer of the initial state to the state $|\psi\rangle_5$ that occurs only at $t=t_{iSWAP}$.

3 Square root swap gates

The situation is more complicated for realization of \sqrt{iSWAP} gate because it is impossible to eliminate state $|\psi\rangle_5$ with evolution based on matrix (12). Below, we propose a universal mechanism for *collective dynamical elimination* (CDE –procedure) of the state $|\psi\rangle_5$ for realization of \sqrt{iSWAP} gate by using the multi-atomic ensemble encoding for single qubits and cavity mediated collective interaction.

Scheme of spatial arrangement of the processing nodes and cavities for realization of the \sqrt{iSWAP} is presented in Fig. 2. Here, we insert the 1-nd and 2-rd nodes in two different single mode QED cavities characterized by high quality factors for π -modes. We assume that each π -mode interacts only with the atoms of one node and is decoupled from the basic cavity field mode that is possible for large enough spectral detuning of the local QED cavity modes.

Thus, we take the additional field Hamiltonians $H_{r\pi} = \sum_m^{1,2} \hbar \omega_{k_0} a_{k_0\pi_m}^+ a_{k_0\pi_m}$ and interaction of photons with the atoms in the 1st and the 2nd nodes $H_{r-a}^{(\pi)} = \sum_m^{1,2} \sum_{j_m} \left(g_{k_0\pi_m}^{(m)} e^{i\vec{k}_0 \vec{r}_j} S_{j_m}^+ a_{k_0\pi_m} + g_{k_0\pi_m}^{(m)*} e^{-i\vec{k}_0 \vec{r}_j} S_{j_m}^- a_{k_0\pi_m}^+ \right).$

By assuming a large enough spectral detuning of atomic frequencies from the field mode and absence of real photons in the QED cavities we find the following effective Hamiltonian similar to previous section

Figure 2: Layout for realization of the two-qubit $\sqrt{i\text{SWAP}}$ gates (σ is the mode of common cavity; π_1 and π_2 are the local modes).

$$\begin{aligned}
H_s = & \sum_m^{1,2} \sum_{j_m} \hbar \omega_m S_{j_m}^z + \sum_m^{1,2} \sum_{i_m j_m} \frac{|g_{k_0 \sigma}^{(m)}|^2}{\hbar \Delta'_m} e^{i \vec{k}_0 \vec{r}_{i_m j_m}} S_{i_m}^+ S_{j_m}^- + \sum_m^{1,2} \sum_{i_m j_m} \frac{|g_{k_0 \pi}^{(1)}|^2}{\hbar \Delta'_m} e^{i \vec{k}_0 \vec{r}_{i_m j_m}} S_{i_m}^+ S_{j_m}^- + \\
& + \frac{1}{2\hbar} \left(\frac{1}{\Delta_1} + \frac{1}{\Delta_2} \right) \sum_{j_1 j_2} \left(g_{k_0 \sigma}^{(1)} g_{k_0 \sigma}^{(2)*} e^{i \vec{k}_0 \vec{r}_{j_1 j_2}} S_{j_1}^+ S_{j_2}^- + g_{k_0 \sigma}^{(1)*} g_{k_0 \sigma}^{(2)} e^{-i \vec{k}_0 \vec{r}_{j_1 j_2}} S_{j_2}^- S_{j_1}^+ \right), \quad (12)
\end{aligned}$$

where $\Delta_{1,2} = \omega_{1,2} - \omega_o$ are the atomic frequency detunings from the common cavity mode and $\Delta'_{1,2} = \omega_{1,2} - \omega_{k_o}$ are the atomic detunings from the frequency of the local QED cavities having the same frequency ω_{k_o} . To be concrete, we take below $\Delta'_{1,2} = -\Delta_{1,2} = -\Delta$, $\Delta > 0$.

The second and third terms in Eq. (12) describes the atom-atom interactions inside each node via the exchange of σ and π virtual photons, while the last term describes the interaction due to the exchange of virtual σ photons between the atoms situating in different nodes. Again by assuming equal number of atoms in the two nodes $N_1 = N_2 = N$, we get the following matrix representation for the new effective Hamiltonian \hat{H}_{eff} in the basis of the five states

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \Omega_s N & \Omega_\sigma N & 0 & 0 \\ 0 & \Omega_\sigma N & \Omega_s N & 0 & 0 \\ 0 & 0 & 0 & 2\Omega_s N & 2\Omega_\sigma \sqrt{N(N-1)} \\ 0 & 0 & 0 & 2\Omega_\sigma \sqrt{N(N-1)} & 2\Omega_s(N-1) \end{pmatrix}, \quad (13)$$

where $\Omega_s = \Omega_\sigma + \Omega_\pi$, $\Omega_\sigma = |g_\sigma|^2/\Delta$, $\Omega_\pi = -|g_\pi|^2/\Delta$.

For the initial state (1.1), the atomic wave function evolves as follows

$$\begin{aligned}
\Psi_2(t) = & \alpha_1 \alpha_2 \psi_1 \\
& + \exp[-i\Omega_s N t] \{ \beta_1 \alpha_2 [\cos(\Omega_\sigma N t) \psi_2 - i \sin(\Omega_\sigma N t) \psi_3] + \alpha_1 \beta_2 [\cos(\Omega_\sigma N t) \psi_3 - i \sin(\Omega_\sigma N t) \psi_2] \} \\
& + \exp[-i\Omega_s (2N-1)t] \beta_1 \beta_2 \{ [\cos(St) - i \frac{\Omega_s}{S} \sin(St)] \psi_4 - i \frac{2\Omega_\sigma \sqrt{N(N-1)}}{S} \sin(St) \psi_5 \}, \quad (14)
\end{aligned}$$

where $S = \sqrt{4\Omega_\sigma^2 N(N-1) + \Omega_s^2}$.

We choose the following parameters for the evolution of Eq. (14) providing the dynamical elimination of the state ψ_5 :

$$\begin{aligned} 1) \Omega_\sigma Nt &= \pi(\frac{1}{4} + \frac{1}{2}\mu + n); \mu = 0, 1; n = 0, 1, \dots, \\ 2) St &= \pi k, k = 1, 2, \dots, \end{aligned} \quad (15)$$

that leads to the following entangled state of the nodes

$$\begin{aligned} \Psi_2(t) &= \alpha_1 \alpha_2 \psi_1 \\ &+ (-1)^n \frac{1}{\sqrt{2}} \exp[-i\Omega_s Nt] \{ [(-1)^\mu \beta_1 \alpha_2 - i\alpha_1 \beta_2] \psi_2 + [(-1)^\mu \alpha_1 \beta_2 - i\beta_1 \alpha_2] \psi_3 \} \\ &+ (-1)^k \exp[-i\Omega_s (2N-1)t] \beta_1 \beta_2 \psi_4, \end{aligned} \quad (16)$$

where Ω_s is determined by the two conditions (15). In particular we write three sets of parameters for possible realizations of CDE procedure characterized by weaker coupling of atoms with σ -mode ($n=0,1$; $\mu=0,1$):

- 1) $n = 0, \mu = 0, k = 1 : \Omega_\sigma Nt = \pi/4, St = \pi \rightarrow |\Omega_s|t = \sqrt{3}\pi, \frac{|\Omega_s|}{\Omega_\sigma N} = 4\sqrt{3} \approx 6.92$;
- 2) $n = 0, \mu = 1, k = 2 : \Omega_\sigma Nt = 3\pi/4, St = 2\pi \rightarrow |\Omega_s|t = \sqrt{7}\pi, \frac{|\Omega_s|}{\Omega_\sigma N} = \frac{4\sqrt{7}}{3} \approx 5.53$;
- 3) $n = 1, \mu = 0, k = 3 : \Omega_\sigma Nt = 5\pi/4, St = 3\pi \rightarrow |\Omega_s|t = \sqrt{11}\pi, \frac{|\Omega_s|}{\Omega_\sigma N} = \frac{4\sqrt{11}}{5} \approx 2.65$

and so on.

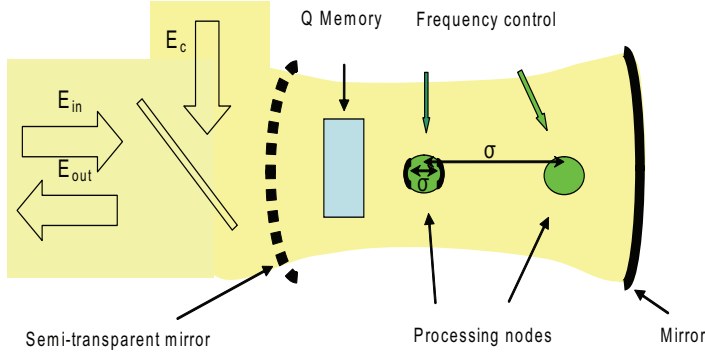
Another interesting case occurs for stronger coupling of the atoms with local π -modes of the QED cavities when $|\Omega_\pi| \gg N\Omega_\sigma$. Here, we get a *collective blockade* of state ψ_5 that provides the following atomic evolution

$$\begin{aligned} \Psi_2(t) &= \alpha_2 \alpha_3 \psi_1 + \exp[-i\Omega_s Nt] \{ \beta_2 \alpha_3 [\cos(\Omega_\sigma Nt) \psi_2 - i \sin(\Omega_\sigma Nt) \psi_3] \\ &+ \alpha_2 \beta_3 [\cos(\Omega_\sigma Nt) \psi_3 - i \sin(\Omega_\sigma Nt) \psi_2] \} \\ &+ \exp[-i2\Omega_s Nt] \beta_2 \beta_3 \psi_4, \end{aligned} \quad (17)$$

yielding the entangled state of the two nodes if only the condition (15) is satisfied. So here, we can vary the coupling constant Ω_σ and interaction time t in some possible intervals providing a realization of general iSWAP gate with arbitrary tunable angle of rotation $\Omega_\sigma Nt$. *Collective blockade* needs more quality micro-cavities than *collective dynamical elimination technique* but it is more robust being operative for all necessary temporal durations.

4 Controlled swap gates

Let's consider two atomic ensembles situating in two separate nodes in the common resonator as shown in Fig.3. With that, one of these nodes has its own micro-resonator. We can introduce signal and control photons through a beam splitter into the system. Photons are stored for a time in quantum memory situating also in common resonator. After absorption of photons by quantum memory, we raise reflectivity of input-output mirror in order to make resonator perfect. First, signal photon is transferred from quantum memory to one of processing nodes and frequency of atomic transitions in processing nodes is

Figure 3: Scheme of control *SWAP* gate (*CSWAP*) on multi-atomic ensembles in resonator.

tuned out of resonance with the cavity. Then, we release from quantum memory the control photon and detune memory from resonance with it. With that, control photon can not be absorbed by the memory and processing nodes or released from the cavity at these conditions.

Using Hamiltonian (8) and states $\psi_1 = |0\rangle_1 |0\rangle_2$, $\psi_2 = |1\rangle_1 |0\rangle_2$, $\psi_3 = |0\rangle_1 |1\rangle_2$, $\psi_4 = |1\rangle_1 |1\rangle_2$, $\psi_5 = |2\rangle_1 |0\rangle_2$ and $\psi_6 = |0\rangle_1 |2\rangle_2$, we get for total wave function

$$\psi(t) = c_1(t) \psi_1 + c_2(t) \psi_2 + c_3(t) \psi_3 + c_4(t) \psi_4 + c_5(t) \psi_5 + c_6(t) \psi_6, \quad (18)$$

the following Schrödinger equation

$$\begin{aligned} \frac{d\psi}{dt} = & i \frac{N}{2} \{ \omega_1 + \omega_2 + 2n(\Omega_1 + \Omega_2) \} c_1 \psi_1 \\ & + i \left\{ \left(\frac{N}{2} - 1 \right) (\omega_1 + 2n\Omega_1) + \frac{N}{2} (\omega_2 + 2n\Omega_2) - iN\Omega_1 \right\} c_2 \psi_2 - iN\Omega_s c_3 \psi_2 \\ & - iN\Omega_s c_2 \psi_3 + i \left\{ \frac{N}{2} (\omega_1 + 2n\Omega_1) + \left(\frac{N}{2} - 1 \right) (\omega_2 + 2n\Omega_2) - N\Omega_2 \right\} c_3 \psi_3 \\ & + i \left\{ \left(\frac{N}{2} - 1 \right) (\omega_1 + \omega_2 + 2n(\Omega_1 + \Omega_2)) - N(\Omega_1 + \Omega_2) \right\} c_4 \psi_4 - i\Omega_s \sqrt{2N(N-1)} (c_5 + c_6) \psi_4 \\ & - i\Omega_s \sqrt{2N(N-1)} c_4 \psi_5 + i \left(\frac{N}{2} - 2 \right) (\omega_1 + 2n\Omega_1) c_5 \psi_5 \\ & - i\Omega_s \sqrt{2N(N-1)} c_4 \psi_6 + i \left(\frac{N}{2} - 2 \right) (\omega_2 + 2n\Omega_2) c_6 \psi_6, \end{aligned} \quad (19)$$

where we have assumed the mode field is in the state with definite number n of photons, $\Omega_1 = \frac{|g_{k_0}^{(1)}|^2}{\hbar^2 \Delta}$, $\Omega_2 = \frac{|g_{k_0}^{(2)}|^2}{\hbar^2 \Delta}$ and $\Omega_s = \frac{g_{k_0}^{(1)} g_{k_0}^{(2)}}{\hbar^2 \Delta}$. If $g_{k_0}^{(2)} \ll g_{k_0}^{(1)}$. Below we are interested in the case when $\Omega_2 \cong 0$ (second node is characterized by lower quality factor in comparison with the first node factor) and equations for c_2 and c_3 can be written as

$$\frac{dc_2}{dt} = i \left\{ \left(\frac{N}{2} - 1 \right) (\omega_1 + 2n\Omega_1) + \frac{N}{2} \omega_2 - N\Omega_1 \right\} c_2 - iN\Omega_s c_3, \quad (20)$$

$$\frac{dc_3}{dt} = i \left\{ \frac{N}{2} (\omega_1 + 2n\Omega_1) + \left(\frac{N}{2} - 1 \right) \omega_2 \right\} c_3 - iN\Omega_s c_2, \quad (21)$$

or

$$\frac{dc_2}{dt} = \frac{i}{\hbar} E_2 c_2 - iN\Omega_s c_3, \quad (22)$$

$$\frac{dc_3}{dt} = \frac{i}{\hbar} E_3 c_3 - iN\Omega_s c_2, \quad (23)$$

where $E_2 = \left(\frac{N}{2} - 1\right) (\omega_1 + 2n\Omega_1) + \frac{N}{2}\omega_2 - N\Omega_1$, and $E_3 = \frac{N}{2} (\omega_1 + 2n\Omega_1) + \left(\frac{N}{2} - 1\right) \omega_2$. The Equations (22), (23) give the following second order equation

$$\frac{d^2 c_3}{dt^2} - \frac{i}{\hbar} (E_2 + E_3) \frac{dc_3}{dt} - \left(\frac{E_2 E_3}{\hbar^2} - N^2 \Omega_s^2 \right) c_3 = 0, \quad (24)$$

with a solution

$$c_3 = C_1 e^{ir_1 t} + C_2 e^{ir_2 t}, \quad (25)$$

where

$$\begin{aligned} r_{1,2} = & \frac{1}{2\hbar} \left\{ \left[\left(\frac{N}{2} - 1 \right) (\omega_1 + 2n\Omega_1) + \frac{N}{2}\omega_2 - N\Omega_1 \right] + \left[\frac{N}{2} (\omega_1 + 2n\Omega_1) + \left(\frac{N}{2} - 1 \right) \omega_2 \right] \right\} \\ & \pm \sqrt{\frac{1}{4} (\omega_1 - \omega_2 + N\Omega_1 + 2n\Omega_1)^2 + N^2 \Omega_s^2}. \end{aligned} \quad (26)$$

With the initial conditions $c_2 = 1$ and $c_3 = 0$, we have

$$C_1 = -C_2 = -\frac{N\Omega_s}{\sqrt{(\omega_1 - \omega_2 + N\Omega_1 + 2n\Omega_1)^2 + N^2 \Omega_s^2}}, \quad (27)$$

that simplifies at $\omega_1 - \omega_2 + N\Omega_1 = 0$ to

$$C_1 = -C_2 = -\frac{N\Omega_s}{\sqrt{N^2 \Omega_s^2 + 4n^2 \Omega_1^2}}. \quad (28)$$

We see that if $n = 1$ at $2\Omega_1 \gg N\Omega_s$ we have $C_1 = C_2 = c_3 \cong 0$ and if $n = 0$ we have $C_1 = -1$, $C_2 = 1$ and in the first case no swap occurs and in the second case we have swapping solution

$$c_2 = e^{\frac{i}{2\hbar} \left\{ \left(\frac{N}{2} - 1 \right) \omega_1 + \frac{N}{2} \omega_2 - N\Omega_1 \right\} t} \cos(N\Omega_s t), \quad (29)$$

and

$$c_3 = -ie^{\frac{i}{2\hbar} \left\{ \left(\frac{N}{2} - 1 \right) \omega_1 + \frac{N}{2} \omega_2 - N\Omega_1 \right\} t} \sin(N\Omega_s t). \quad (30)$$

In the first case no swap occurs and in the second case we have swapping solution.

5 Summary

So, we have considered *i*SWAP, $\sqrt{i$ SWAP and *C*SWAP gates. *i*SWAP gate can be used for efficient transfer of qubit between various nodes of quantum computer. $\sqrt{i$ SWAP gate which entangles the two qubits provides a complete set of universal quantum gates together with single qubit operations. Here, we note that the single qubit gates can be performed by transfer the atomic qubit to photonic qubit in waveguide where it can be rotated on arbitrary angle by usual optical tools [7]. We can also return the qubit back to QM node on demand as it has been shown above. Another possibility to implement the single qubit gates

is to transfer it to the node with single resonant atom which state can be controlled by external classical field [8]. Also we can mark the principle possibility to exploit the collective blockade mechanism for realization of the single qubit gate similar to approach developed for usual blockade mechanism [3] and exploitation of Raman transition between the collective atomic states [4]. Fast CSWAP gate can be used for efficient realization of promising quantum algorithm of fingerprinting [9]. The proposed protocols of two and three-qubit gates also make a creation of large scale universal quantum computer more feasible with the multi-atomic encoding of the single qubit states.

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